MECANIQUE ANALYTIQUE

Série 6 Hiver 2009-2010 informations: http://cag.epfl.ch
pour SMA

Exercice 1. (**Quaternions**) The division ring $\mathbb{H}$ (or by abuse of language, the non-commutative field) of quaternions is generated over the reals by three elements $\mathbf{i}, \mathbf{j}, \mathbf{k}$ with the relations

$$\begin{aligned}&\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,\\ &\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.\end{aligned}$$

Quaternionic multiplication is performed in the usual manner (like polynomial multiplication) taking the above relations into account. If $a \in \mathbb{H}$, we write

$$a = (a_s, \mathbf{a}_v) = a_s + a_v^1\mathbf{i} + a_v^2\mathbf{j} + a_v^3\mathbf{k}$$

for the *scalar* and *vectorial part of the quaternion*, where $a_s, a_v^1, a_v^2, a_v^3 \in \mathbb{R}$. Quaternions having zero scalar part are also called *pure quaternions*.

(a) Show that with the above notation, quaternionic multiplication has the expression

$$ab = (a_s b_s - \mathbf{a}_v \cdot \mathbf{b}_v, a_s\mathbf{b}_v + b_s\mathbf{a}_v + \mathbf{a}_v \times \mathbf{b}_v).$$

In addition, every quaternion $a = (a_s, \mathbf{a}_v)$ has a conjugate $\bar{a} = (a_s, -\mathbf{a}_v)$, that is, the real numbers are fixed by the conjugation and $\bar{\mathbf{i}} = -\mathbf{i}$, $\bar{\mathbf{j}} = -\mathbf{j}$, $\bar{\mathbf{k}} = -\mathbf{k}$.

(b) Show that $\overline{ab} = \bar{b}\bar{a}$ and that every quaternion $a \neq 0$ has an inverse given by $a^{-1} = \bar{a}/|a|^2$, where

$$|a|^2 := a\bar{a} = \bar{a}a = a_s^2 + ||\mathbf{a}_v||^2.$$

(c) Show that the unit quaternions $S^3 := \{\, a \in \mathbb{H} \mid |a| = 1\}$ form a Lie group that is isomorphic to SU(2).

(d) Show that the Lie Algebra of S^3 is isomorphic to the pure quaternions $\mathbb{R}^3$ and obtain explicit formulas for the adjoint action and the Lie bracket.

(e) Give an explicit expression for a 2 : 1 Lie group homomorphism $S^3 \to \mathrm{SO}(3)$.
(Hint: Use the Lie group homomorphism $\pi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ and the Lie group isomorphism between S^3 and SU(2) found in part (c)).

Corrigé exercice 1.

(a) and (b) Follow by a simple direct calculation.

(c) Define the map,

$$a = a_s + a_v^1\mathbf{i} + a_v^2\mathbf{j} + a_v^3\mathbf{k} \in S^3 \mapsto \begin{pmatrix} a_s - ia_v^3 & -a_v^2 - ia_v^1 \\ a_v^2 - ia_v^1 & a_s + ia_v^3 \end{pmatrix} \in \mathrm{SU}(2).$$

It is clear that it is a diffeomorphism. Moreover, a direct calculation shows that it is a group homomorphism and therefore a Lie group isomorphism.

(d) Since the Lie algebra of S^3 is the tangent space at 1, it follows that it is isomorphic to the pure quaternions $\mathbb{R}^3$. We begin by determining the adjoint action of S^3 on its Lie algebra.

If $a \in S^3$ and $\mathbf{b}_v$ is a pure quaternion, the derivative of conjugation is given by

$$\begin{aligned}\operatorname{Ad}_a \mathbf{b}_v = a\mathbf{b}_v a^{-1} = a\mathbf{b}_v \frac{\bar{a}}{|a|^2} &= \frac{1}{|a|^2}(-\mathbf{a}_v \cdot \mathbf{b}_v \,,\, a_s\mathbf{b}_v + \mathbf{a}_v \times \mathbf{b}_v)(a_s, -\mathbf{a}_v) \\ &= \frac{1}{|a|^2}(\,0\,,\, 2a_s(\mathbf{a}_v \times \mathbf{b}_v) + 2(\mathbf{a}_v \cdot \mathbf{b}_v)\mathbf{a}_v + (a_s^2 - ||\mathbf{a}_v||^2)\mathbf{b}_v).\end{aligned}$$

Therefore, if $a(t) = (a_s(t), \mathbf{a}_v(t)) \in S^3$ is a smooth curve such that $a(0) = (a_s(0), \mathbf{a}_v(0)) = (1, \mathbf{0})$ and $a'(0) = (a_s'(0), \mathbf{a}_v'(0)) = (0, \mathbf{a}_v)$, the Lie bracket $[\mathbf{a}_v, \mathbf{b}_v]$ of the pure quaternions $\mathbf{a}_v, \mathbf{b}_v \in \mathbb{R}^3$ is given by

$$\begin{aligned}[\mathbf{a}_v, \mathbf{b}_v] &= \left.\frac{d}{dt}\right|_{t=0} \operatorname{Ad}_{a(t)} \mathbf{b}_v \\ &= \left.\frac{d}{dt}\right|_{t=0} \frac{1}{|a(t)|^2} \left(0, 2a_s(t)(\mathbf{a}_v(t) \times \mathbf{b}_v) + 2(\mathbf{a}_v(t) \cdot \mathbf{b}_v)\mathbf{a}_v(t) + \left(a_s(t)^2 - \|\mathbf{a}_v(t)\|^2\right) \mathbf{b}_v\right) \\ &= (0, 2(\mathbf{a}_v \times \mathbf{b}_v))\end{aligned}$$

Thus, *the Lie Algebra of S^3 is $\mathbb{R}^3$ relative to the Lie bracket given by twice the cross product of vectors.*

(e) As in part (c) we associate the matrix

$$U = \begin{pmatrix} a_s - ia_v^3 & -a_v^2 - ia_v^1 \\ a_v^2 - ia_v^1 & a_s + ia_v^3 \end{pmatrix} \in \mathrm{SU}(2).$$

to the unit quaternion $a = a_s + a_v^1\mathbf{i} + a_v^2\mathbf{j} + a_v^3\mathbf{k} \in S^3$. The rotation matrix $A = \pi(U) \in \mathrm{SO}(3)$ is defined by the condition

$$\begin{aligned}(A\mathbf{x}) \cdot \boldsymbol{\sigma} &= (\pi(U)\mathbf{x}) \cdot \boldsymbol{\sigma} = U(\mathbf{x} \cdot \boldsymbol{\sigma})U^\dagger \\ &= \begin{pmatrix} a_s - ia_v^3 & -a_v^2 - ia_v^1 \\ a_v^2 - ia_v^1 & a_s + ia_v^3 \end{pmatrix} \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^3 \end{pmatrix} \\ &\times \begin{pmatrix} a_s + ia_v^3 & a_v^2 + ia_v^1 \\ -a_v^2 + ia_v^1 & a_s - ia_v^3 \end{pmatrix} \\ &= [(a_s^2 + (a_v^1)^2 - (a_v^2)^2 - (a_v^3)^2)x^1 + 2(a_v^1a_v^2 - a_sa_v^3)x^2 + 2(a_sa_v^2 + a_v^1a_v^3)x^3]\sigma_1 \\ &\qquad + [2(a_v^1a_v^2 - a_sa_v^3)x^1 + (a_s^2 - (a_v^1)^2 + (a_v^2)^2 - (a_v^3)^2)x^2 + 2(a_v^2a_v^3 - a_sa_v^1)x^3]\sigma_2 \\ &\qquad + [2(a_v^1a_v^3 - a_sa_v^2)x^1 + +2(a_s^2a_v^1 + a_v^2a_v^3)x^2 + (a_s^2 - (a_v^1)^2 - (a_v^2)^2 + (a_v^3)^2)x^3]\sigma_3.\end{aligned}$$

Thus, taking into account that $a_s^2 + (a_v^1)^2 + (a_v^2)^2 + (a_v^3)^2 = 1$, we find that the expression for the matrix A is

$$\begin{aligned}&\begin{pmatrix} 2a_s^2 + 2(a_v^1)^2 - 1 & 2(-a_sa_v^3 + a_v^1a_v^2) & 2(a_sa_v^2 + a_v^1a_v^3) \\ 2(a_sa_v^3 + a_v^1a_v^2 & 2a_s^2 + 2(a_v^2)^2 - 1 & 2(-a_sa_v^1 + a_v^2a_v^3) \\ 2(-a_sa_v^2 + a_v^1a_v^3) & 2(a_sa_v^1 + a_v^2a_v^3) & 2a_s^2 + 2(a_v^3)^2 - 1 \end{pmatrix} \\ &\qquad\qquad = (2a_s^2 - 1)I + 2a_s\hat{\mathbf{a}}_v + 2\mathbf{a}_v \otimes \mathbf{a}_v,\end{aligned}$$

where $\mathbf{a}_v \otimes \mathbf{a}_v$ is the symmetric matrix whose (i, j) entry equals $a_v^ia_v^j$. The map

$$a \in S^3 \mapsto (2a_s^2 - 1)I + 2a_s\hat{\mathbf{a}}_v + 2\mathbf{a}_v \otimes \mathbf{a}_v \in \mathrm{SO}(3)$$

is called the *Euler-Rodrigues parametrization.*

Exercice 2. Show that the 2 to 1 Lie group homomorphism $\pi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ has the following explicit expression

$$\pi\left(\begin{bmatrix}\alpha & -\bar{\beta}\\ \beta & \bar{\alpha}\end{bmatrix}\right) = \begin{bmatrix}\operatorname{Re}(\alpha^2-\beta^2) & \operatorname{Im}(\alpha^2-\beta^2) & 2\operatorname{Re}(\bar{\alpha}\beta)\\ -\operatorname{Im}(\alpha^2+\beta^2) & \operatorname{Re}(\alpha^2+\beta^2) & 2\operatorname{Im}(\bar{\alpha}\beta)\\ -2\operatorname{Re}(\alpha\beta) & -2\operatorname{Im}(\alpha\beta) & |\alpha|^2-|\beta|^2\end{bmatrix},$$

where $\alpha, \beta \in \mathbb{C}$ satisfy $|\alpha|^2 + |\beta|^2 = 1$.

Corrigé exercice 2. Motivated by the map of S^3 to $\mathrm{SU}(2)$, write an element of $\mathrm{SU}(2)$ as follows

$$A = \begin{bmatrix}\alpha & -\bar{\beta}\\ \beta & \bar{\alpha}\end{bmatrix} := \begin{bmatrix}y^0 - iy^3 & -y^2 - iy^1\\ y^2 - iy^1 & y^0 + iy^3\end{bmatrix}$$

with $|\alpha|^2 + |\beta|^2 = (y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 = 1$. The adjoint of A is given by

$$A^\dagger = \begin{bmatrix}y^0 + iy^3 & y^2 + iy^1\\ -y^2 + iy^1 & y^0 - iy^3\end{bmatrix}.$$

For any $\mathbf{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$ we have

$$\begin{aligned}\mathbf{x}\cdot\boldsymbol{\sigma} &= x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3\\ &= x^1\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + x^2\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + x^3\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\\ &= \begin{bmatrix}x^3 & x^1 - ix^2\\ x^1 + ix^2 & -x^3\end{bmatrix}.\end{aligned}$$

A straightforward (but lengthy) direct computation gives the following entries for $A(\mathbf{x}\cdot\boldsymbol{\sigma})A^\dagger$:

$$\begin{aligned}\left(A(\mathbf{x}\cdot\boldsymbol{\sigma})A^\dagger\right)_{11} &= -\left(A(\mathbf{x}\cdot\boldsymbol{\sigma})A^\dagger\right)_{22}\\ &= 2x^1\left(-y^0y^2 + y^1y^3\right) + 2x^2\left(y^0y^1 + y^2y^3\right)\\ &\quad + x^3\left((y^0)^2 + (y^3)^2 - (y^1)^2 - (y^2)^2\right)\end{aligned}$$

$$\begin{aligned}\left(A(\mathbf{x}\cdot\boldsymbol{\sigma})A^\dagger\right)_{12} &= \overline{\left(A(\mathbf{x}\cdot\boldsymbol{\sigma})A^\dagger\right)_{21}}\\ &= x^1\left((y^0)^2 - (y^3)^2 + (y^1)^2 - (y^2)^2\right) + 2x^2\left(y^1y^2 - y^0y^3\right)\\ &\quad + 2x^3\left(y^0y^2 + y^1y^3\right)\\ &\quad - i\left[2x^1\left(y^1y^2 + y^0y^3\right) + x^2\left((y^0)^2 - (y^3)^2 - (y^1)^2 + (y^2)^2\right)\right.\\ &\qquad \left.+2x^3\left(y^2y^3 - y^0y^1\right)\right]\end{aligned}$$

and hence

$$\begin{aligned}&A(\mathbf{x}\cdot\boldsymbol{\sigma})A^\dagger =\\ &\quad\left[x^1\left((y^0)^2 - (y^3)^2 + (y^1)^2 - (y^2)^2\right) + 2x^2\left(y^1y^2 - y^0y^3\right)\right.\\ &\qquad\quad \left.+2x^3\left(y^0y^2 + y^1y^3\right)\right]\sigma_1\\ &\quad + \left[2x^1\left(y^1y^2 + y^0y^3\right) + x^2\left((y^0)^2 - (y^3)^2 - (y^1)^2 + (y^2)^2\right)\right.\\ &\qquad\quad \left.+2x^3\left(y^2y^3 - y^0y^1\right)\right]\sigma_2\\ &\quad + \left[2x^1\left(-y^0y^2 + y^1y^3\right) + 2x^2\left(y^0y^1 + y^2y^3\right)\right.\\ &\qquad\quad \left.+x^3\left((y^0)^2 + (y^3)^2 - (y^1)^2 - (y^2)^2\right)\right]\sigma_3.\end{aligned}$$

From the formula $(\pi(A)\mathbf{x}) \cdot \boldsymbol{\sigma} = A(\mathbf{x} \cdot \boldsymbol{\sigma})A^{-1}$ in the text, we conclude that

$$\begin{aligned}
&\pi(A)\mathbf{x} = \\
&\begin{bmatrix} x^1\left((y^0)^2-(y^3)^2+(y^1)^2-(y^2)^2\right)+2x^2\left(y^1y^2-y^0y^3\right)+2x^3\left(y^0y^2+y^1y^3\right) \\ 2x^1\left(y^1y^2+y^0y^3\right)+x^2\left((y^0)^2-(y^3)^2-(y^1)^2+(y^2)^2\right)+2x^3\left(y^2y^3-y^0y^1\right) \\ 2x^1\left(-y^0y^2+y^1y^3\right)+2x^2\left(y^0y^1+y^2y^3\right)+x^3\left((y^0)^2+(y^3)^2-(y^1)^2-(y^2)^2\right) \end{bmatrix} \\
&= B\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}.
\end{aligned}$$

where B is the matrix

$$\begin{bmatrix} (y^0)^2-(y^3)^2+(y^1)^2-(y^2)^2 & 2\left(y^1y^2-y^0y^3\right) & 2\left(y^0y^2+y^1y^3\right) \\ 2\left(y^1y^2+y^0y^3\right) & (y^0)^2-(y^3)^2-(y^1)^2+(y^2)^2 & 2\left(y^2y^3-y^0y^1\right) \\ 2\left(-y^0y^2+y^1y^3\right) & 2\left(y^0y^1+y^2y^3\right) & (y^0)^2+(y^3)^2-(y^1)^2-(y^2)^2 \end{bmatrix}.$$

As indicated at the beginning of this solution, we have $\alpha := y^0 - iy^3$ and $\beta := y^2 - iy^1$. Note that

$$\begin{aligned}
|\alpha|^2 &= (y^0)^2+(y^3)^2, & |\beta|^2 &= (y^2)^2+(y^1)^2, \\
\alpha^2 &= (y^0)^2-(y^3)^2-2iy^0y^3, & \beta^2 &= (y^2)^2-(y^1)^2-2iy^1y^2 \\
\alpha\beta &= y^0y^2-y^1y^3+i(-y^2y^3-y^0y^1), & \overline{\alpha}\beta &= y^0y^2+y^1y^3+i(y^2y^3-y^0y^1)
\end{aligned}$$

and therefore

$$\begin{aligned}
\operatorname{Re}(\alpha^2-\beta^2) &= (y^0)^2-(y^3)^2+(y^1)^2-(y^2)^2, \\
\operatorname{Im}(\alpha^2-\beta^2) &= y^1y^2-y^0y^3 \\
-\operatorname{Im}(\alpha^2+\beta^2) &= y^1y^2+y^0y^3, \\
\operatorname{Re}(\alpha^2+\beta^2) &= (y^0)^2-(y^3)^2-(y^1)^2+(y^2)^2 \\
-2\operatorname{Re}(\alpha\beta) &= 2(-y^0y^2+y^1y^3), \\
-2\operatorname{Im}(\alpha\beta) &= 2(y^2y^3+y^0y^1) \\
2\operatorname{Re}(\bar{\alpha}\beta) &= 2(y^0y^2+y^1y^3), \\
2\operatorname{Im}(\bar{\alpha}\beta) &= 2(y^2y^3-y^0y^1) \\
|\alpha|^2-|\beta|^2 &= (y^0)^2+(y^3)^2-(y^1)^2-(y^2)^2,
\end{aligned}$$

which proves the formula.

Exercice 3. This exercise concerns *Euler angles*, that are important in the context of rigid body mechanics.

(a) Show that the map

$$(\theta, \varphi, \psi) \in \,]0, \pi[\, \times [0, 2\pi[\, \times [-2\pi, 2\pi[\quad \longrightarrow$$
$$U := \left\{ \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} \in \mathrm{SU}(2) \,\middle|\, \alpha\beta \neq 0 \right\}$$

given by $\alpha = \cos\frac{\theta}{2} e^{i\frac{\varphi+\psi}{2}}$, $\beta = \sin\frac{\theta}{2} e^{i\frac{\varphi-\psi+\pi}{2}}$ is a bijection. Thus, every element of SU(2) with $\alpha\beta \neq 0$ can be uniquely written as

$$A(\varphi, \theta, \psi) := \begin{bmatrix} \cos\frac{\theta}{2} e^{i\frac{\varphi+\psi}{2}} & i\sin\frac{\theta}{2} e^{-i\frac{\varphi-\psi}{2}} \\ i\sin\frac{\theta}{2} e^{i\frac{\varphi-\psi}{2}} & \cos\frac{\theta}{2} e^{-i\frac{\varphi+\psi}{2}} \end{bmatrix}$$

for $(\varphi, \theta, \psi) \in [0, 2\pi[\, \times \,]0, \pi[\, \times [-2\pi, 2\pi[$.

(b) Use the preceding exercise to show that the 2 to 1 Lie group homomorphism $\pi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ restricted to this open subset $U \subset \mathrm{SU}(2)$ is given by

$$\pi\left(\begin{bmatrix} \cos\frac{\theta}{2} e^{i\frac{\varphi+\psi}{2}} & i\sin\frac{\theta}{2} e^{-i\frac{\varphi-\psi}{2}} \\ i\sin\frac{\theta}{2} e^{i\frac{\varphi-\psi}{2}} & \cos\frac{\theta}{2} e^{-i\frac{\varphi+\psi}{2}} \end{bmatrix}\right)$$
$$= \begin{bmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi & \sin\theta\sin\psi \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{bmatrix}.$$

The angles (φ, θ, ψ) appearing in these representations of elements of SU(2) and SO(3) are called *Euler angles*. It is possible to give a geometrical interpretation of the Euler angles for elements of SO(3) in terms of three successive rotations. For elements of SO(3) we have $(\theta, \varphi, \psi) \in]0, \pi[\, \times [0, 2\pi[\, \times [0, 2\pi[$.

Corrigé exercice 3. Since SU(2) is diffeomorphic to

$$S^3 = \{(\alpha, \beta) \in \mathbb{C}^2 \mid |\alpha|^2 + |\beta|^2 = 1\}$$

we will show that the map given in the statement is a bijection onto $V := \{(\alpha, \beta) \in S^3 \mid \alpha\beta \neq 0\}$. The condition $|\alpha|^2 + |\beta|^2 = 1$ implies that $|\alpha|$ is arbitrary in the interval $[0, 1]$. Thus it can be uniquely written as $\cos\frac{\theta}{2}$ for $0 \leq \theta \leq \pi$. Then $|\beta| = \sin\frac{\theta}{2}$. Since $\alpha\beta \neq 0$, we must have $\theta \neq 0, \pi$ and hence $0 < \theta < \pi$. Therefore $\alpha = \cos\frac{\theta}{2} e^{iA}$ and $\beta = \sin\frac{\theta}{2} e^{iB}$, where $A, B \in \mathbb{R}$. Now write

$$A = \frac{\varphi + \psi}{2} \quad \text{and} \quad B = \frac{\pi + \varphi - \psi}{2},$$

which is equivalent to

$$\varphi = A + B - \frac{\pi}{2} \quad \text{and} \quad \psi = A - B + \frac{\pi}{2}.$$

Thus it remains to show that the map

$$(\varphi, \psi) \in [0, 2\pi[\, \times [-2\pi, 2\pi[\longmapsto \left(e^{i\frac{\varphi+\psi}{2}}, e^{i\frac{\pi+\varphi-\psi}{2}}\right) \in S^1 \times S^1$$

is bijective. Take $A, B \in [0, 2\pi[$ to describe uniquely (e^{iA}, e^{iB}). Then $0 \leq A + B - \frac{\pi}{2} < \frac{5\pi}{2}$, which is an interval of length larger than 2π, so if we take φ to be $A + B - \frac{\pi}{2}$ modulo 2π, we get an angle in the interval $[0, 2\pi[$. Now take $A \in [0, 2\pi[$ and $B \in]0, 2\pi]$ to uniquely describe (e^{iA}, e^{iB}). Then

$-\frac{3\pi}{2} \le A - B + \frac{\pi}{2} < \frac{5\pi}{2}$, which is an interval of length 4π, so we can take ψ to be $A - B + \frac{\pi}{2}$ modulo 4π to get an angle in the interval $[-2\pi, 2\pi[$. This shows that the map is surjective.

To show injectivity, let

$$e^{i\frac{\varphi+\psi}{2}} = e^{i\frac{\varphi'+\psi'}{2}} \quad \text{and} \quad e^{i\frac{\pi+\varphi-\psi}{2}} = e^{i\frac{\pi+\varphi'-\psi'}{2}},$$

which is equivalent to

$$\frac{\varphi+\psi}{2} - \frac{\varphi'+\psi'}{2} = 2k\pi \quad \text{and} \quad \frac{\varphi-\psi}{2} - \frac{\varphi'-\psi'}{2} = 2l\pi,$$

for some $k, l \in \mathbb{Z}$. Therefore $\varphi + \psi - \varphi' - \psi' = 4k\pi$ and $\varphi - \psi - \varphi' + \psi' = 4l\pi$ which implies $\varphi - \varphi' = 2(k+l)\pi$ and $\psi - \psi' = 2(k-l)\pi$. But since both $\varphi, \varphi' \in [0, 2\pi[$ and $\varphi = \varphi' + 2(k+l)\pi$, it follows that $k = -l$ and hence $\varphi = \varphi'$. But then $\psi = \psi' + 4k\pi$ and since both $\psi, \psi' \in [-2\pi, 2\pi[$ it follows that $k = 0$ and hence $\psi = \psi'$. Thus the map is also injective.

(b) In the notations of the preceding Exercise, we have $\alpha = \cos\frac{\theta}{2} e^{i\frac{\varphi+\psi}{2}}$ and $\beta = i\sin\frac{\theta}{2} e^{i\frac{\varphi-\psi}{2}}$. Hence

$$\begin{array}{ll} |\alpha|^2 = \cos^2\frac{\theta}{2} & |\beta|^2 = \sin^2\frac{\theta}{2} \\ \alpha^2 = \cos^2\frac{\theta}{2} e^{i(\varphi+\psi)} & \beta^2 = -\sin^2\frac{\theta}{2} e^{i(\varphi-\psi)} \\ \alpha\beta = i\sin\frac{\theta}{2}\cos\frac{\theta}{2} e^{i\varphi} & \overline{\alpha}\beta = i\sin\frac{\theta}{2}\cos\frac{\theta}{2} e^{-i\psi} \end{array}$$

and hence

$$\begin{aligned} \operatorname{Re}(\alpha^2 - \beta^2) &= \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi, \\ \operatorname{Im}(\alpha^2 - \beta^2) &= \cos\psi\sin\varphi + \cos\theta\cos\varphi\sin\psi \\ -\operatorname{Im}(\alpha^2 + \beta^2) &= -\sin\psi\cos\varphi - \cos\theta\sin\varphi\cos\psi, \\ \operatorname{Re}(\alpha^2 + \beta^2) &= -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi \\ -2\operatorname{Re}(\alpha\beta) &= \sin\theta\sin\varphi, \\ -2\operatorname{Im}(\alpha\beta) &= -\sin\theta\cos\varphi \\ 2\operatorname{Re}(\bar{\alpha}\beta) &= \sin\theta\sin\psi, \\ 2\operatorname{Im}(\bar{\alpha}\beta) &= \sin\theta\cos\psi \\ |\alpha|^2 - |\beta|^2 &= \cos\theta \end{aligned}$$

which proves the formula for the homomrphism π.