MECANIQUE ANALYTIQUE

Série 2 Hiver 2009-2010 informations: http://cag.epfl.ch
pour SMA

Exercice 1. Show that the bracket of two locally Hamiltonian vector fields on a symplectic manifold (P, Ω) is globally Hamiltonian.

Corrigé exercice 1. Let X, Y be locally Hamiltonian, that is, $\mathbf{d}\mathbf{i}_X\Omega = 0, \mathbf{d}\mathbf{i}_Y\Omega = 0$. We will show that

$$[X, Y] = X_{-\Omega(X,Y)}.$$

We have

$$\begin{aligned}
\mathbf{i}_{[X,Y]}\Omega &= (\pounds_X \mathbf{i}_Y - \mathbf{i}_Y \pounds_X)\Omega \\
&= \mathbf{d}\mathbf{i}_X\mathbf{i}_Y\Omega + \mathbf{i}_X\mathbf{d}\mathbf{i}_Y\Omega - \mathbf{i}_Y\mathbf{d}\mathbf{i}_X\Omega - \mathbf{i}_Y\mathbf{i}_X\mathbf{d}\Omega \\
&= \mathbf{d}(-\Omega(X, Y)).
\end{aligned}$$

We conclude that $[X, Y]$ is globally Hamiltonian with the Hamiltonian function $H = -\Omega(X, Y)$, as required.

Exercice 2. Let X be a constant nonzero vector field on the two-torus. Show that X is locally Hamiltonian but is not globally Hamiltonian.

Corrigé exercice 2. Let (θ_1, θ_2) be the standard 2π-periodic coordinate functions on $\mathbb{T}^2$; recall that in these coordinates, the symplectic form is $\Omega = \mathbf{d}\theta_1 \wedge \mathbf{d}\theta_2$.

There are constants a and b such that $X = a\partial/\partial\theta_1 + b\partial/\partial\theta_2$. We see that X is locally Hamiltonian since $\mathbf{d}\mathbf{i}_X\Omega = \mathbf{d}(a\mathbf{d}\theta_2 - b\mathbf{d}\theta_1) = 0$.

However, X is not globally Hamiltonian. To see this, assume that there exists a function $f : \mathbb{T}^2 \to \mathbb{R}$ such that $\mathbf{d}f = \mathbf{i}_X\Omega$. It follows that $\mathbf{d}f = \mathbf{i}_X\Omega = a\mathbf{d}\theta_2 - b\mathbf{d}\theta_1$ is a constant one-form on $\mathbb{T}^2$. Since $\mathbb{T}^2$ is compact and any continuous function on a compact set achieves its maximum and minimum, there is be a point at which $\mathbf{d}f = 0$. Hence, as $\mathbf{d}f$ is constant, this constant must be zero and thus X is zero, a contradiction.

Exercice 3. Give an example of a symplectic manifold (P, Ω), where Ω is exact, but P is *not* a cotangent bundle.

Corrigé exercice 3. Choose, for example, $(P, \Omega) = (B^2, dx \wedge dy)$, where $B^2 \subset \mathbb{R}^2$ is the open unit disk. Since $dx \wedge dy = d(xdy)$ holds globally on the disk, the symplectic form is exact. But any cotangent bundle clearly has infinite volume, and yet $(B^2, dx \wedge dy)$ has a finite volume, namely π. Thus, no symplectic diffeomorphism between $(B^2, dx \wedge dy)$ and a cotangent bundle can exist.

Exercice 4. Let Ω be a nondegenerate two-form on a manifold P. Form Hamiltonian vector fields and the Poisson bracket using the same definitions as in the symplectic case. Show that Jacobi's identity holds if and only if the two-form Ω is closed.

Corrigé exercice 4. We know that Jacobi's identity holds if Ω is closed, so we establish the converse. Let X_1, X_2, X_3 be Hamiltonian vector fields with Hamiltonian functions f_1, f_2, f_3. Using the properties of the exterior derivative, and the definition of the Poisson bracket, we get

$$\begin{aligned}
\mathbf{d}\Omega(X_1, X_2, X_3) &= X_1[\Omega(X_2, X_3)] - X_2[\Omega(X_1, X_3)] + X_3[\Omega(X_1, X_2)] \\
&\quad - \Omega([X_1, X_2], X_3) + \Omega([X_1, X_3], X_2) - \Omega([X_2, X_3], X_1) \\
&= \{\{f_2, f_3\}, f_1\} + \{f_2, \{f_1, f_3\}\} + \{\{f_1, f_2\}, f_3\} \\
&\quad + \Omega(X_{\{f_1,f_2\}}, X_3) + \Omega(X_{\{f_3,f_1\}}, X_2) + \Omega(X_{\{f_2,f_3\}}, X_1) \\
&= \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} + \{\{f_1, f_2\}, f_3\} \\
&\quad + \{\{f_1, f_2\}, f_3\} + \{\{f_3, f_1\}, f_2\} + \{\{f_2, f_3\}, f_1\} = 0.
\end{aligned}$$

Now, consider any vector fields Y_1, Y_2, Y_3. Since $\mathbf{d}\Omega(Y_1, Y_2, Y_3)$ depends only on the point values of Y_1, Y_2, Y_3, it follows that for each point $z \in P$,

$$\mathbf{d}\Omega(Y_1, Y_2, Y_3)(z) = \mathbf{d}\Omega(X_1, X_2, X_3)(z)$$

for some Hamiltonian vector fields $X_i, i = 1, 2, 3$, such that $X_i(z) = Y_i(z)$. Thus, for all $z \in P$,

$$\mathbf{d}\Omega(Y_1, Y_2, Y_3)(z) = 0.$$

Hence $\mathbf{d}\Omega = 0$, that is, Ω is closed.

Exercice 5. Jacobi–Haretu Coordinates. Consider the three-particle configuration space $Q = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ with elements denoted $\mathbf{r}_1, \mathbf{r}_2$, and $\mathbf{r}_3$. Call the conjugate momenta $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ and equip the phase space T^*Q with the canonical symplectic structure Ω. Let

$$\mathbf{j} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3, \quad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1,$$

and

$$\mathbf{s} = \mathbf{r}_3 - \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2).$$

Show that the form Ω pulled back to the level sets of $\mathbf{j}$ has the form

$$\Omega = d\mathbf{r} \wedge d\pi + d\mathbf{s} \wedge d\sigma,$$

where the variables π and σ are defined by $\pi = \frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_1)$ and $\sigma = \mathbf{p}_3$.

Corrigé exercice 5. We shall write symbolically $\mathbf{dr}_j \wedge \mathbf{dp}_j$ to mean

$$\sum_{i=1}^{3} dr_j^i \wedge dp_{ji}, \qquad j = 1, 2, 3,$$

where $\mathbf{r}_j = (r_j^1, r_j^2, r_j^3)$, and $\mathbf{p}_j = (p_{j1}, p_{j2}, p_{j3})$. The expressions for $\mathbf{r}$ and $\mathbf{s}$ can be solved as functions of $\mathbf{r}_3$ to give

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{r}_3 - \mathbf{s} - \frac{1}{2}\mathbf{r} \\
\mathbf{r}_2 &= \mathbf{r}_3 - \mathbf{s} + \frac{1}{2}\mathbf{r}
\end{aligned}$$

and hence

$$\begin{aligned}\mathbf{dr}_1 \wedge \mathbf{dp}_1 \;&+\; \mathbf{dr}_2 \wedge \mathbf{dp}_2 \,+\, \mathbf{dr}_3 \wedge \mathbf{dp}_3 \\ &= \mathbf{dr}_3 \wedge \mathbf{dp}_1 - \mathbf{ds} \wedge \mathbf{dp}_1 - \frac{1}{2}\mathbf{dr} \wedge \mathbf{dp}_1 + \mathbf{dr}_3 \wedge \mathbf{dp}_2 - \mathbf{ds} \wedge \mathbf{dp}_2 \\ &\quad + \frac{1}{2}\mathbf{dr} \wedge \mathbf{dp}_2 + \mathbf{dr}_3 \wedge \mathbf{dp}_3 \\ &= \mathbf{dr}_3 \wedge \mathbf{d}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) - \mathbf{ds} \wedge \mathbf{d}(\mathbf{p}_1 + \mathbf{p}_2) + \frac{1}{2}\mathbf{dr} \wedge \mathbf{d}(\mathbf{p}_2 - \mathbf{p}_1).\end{aligned}$$

This gives the stated expression of the pulled back form to the level sets of $\mathbf{j}$ since $\mathbf{d}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) = 0$ on this level set.

Exercice 6. Let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a map of the form $\varphi(q,p) = (q, p + \alpha(q))$. Use the canonical one-form $p_i dq^i$ to determine when φ is symplectic.

Corrigé exercice 6. We use the relation $\mathbf{d}(\varphi^*\Theta - \Theta) = 0$, where $\Theta = p_i dq^i$ to find conditions on $\alpha(q)$. Letting

$$\alpha(q^1, \dots q^n) = \alpha_i(q^1, \dots, q^n) dq^i \in \Omega^1(\mathbb{R}^n),$$

we have

$$\varphi^*\Theta = \varphi^*(p_i dq^i) = (p_i + \alpha_i(q^1, \dots, q^n)) dq^i,$$

so

$$\varphi^*\Theta - \Theta = \alpha_i(q^1, \dots, q^n) dq^i = \alpha$$

and hence

$$\mathbf{d}(\varphi^*\Theta - \Theta) = \mathbf{d}\alpha.$$

Thus, φ is symplectic if and only if $\mathbf{d}\alpha = 0$.

Exercice 7. Show how to construct (explicitly) canonical coordinates for the symplectic form $\Omega = f\mu$ on S^2, where μ is the standard area element and where $f : S^2 \to \mathbb{R}$ is a positive function.

Corrigé exercice 7. Let (q,p) denote the desired canonical coordinates on S^2, that is, we write $\Omega = f\mu = dq \wedge dp$. On the other hand, the standard area form on the sphere S^2 of radius 1 is $\mu = \sin\theta d\theta \wedge d\varphi$. Thus we require

$$dq \wedge dp = f(\theta, \varphi) \sin\theta d\theta \wedge d\varphi.$$

Let us seek q and p as functions of φ, θ. Choosing $p = \varphi$, we impose the condition

$$f(\theta, \varphi) \sin\theta = \frac{\partial q}{\partial \theta}(\theta, \varphi),$$

from which we conclude that

$$q(\theta, \varphi) = \int_0^\theta f(\beta, \varphi) \sin\beta \, d\beta$$

is a solution. In addition, the Jacobian of the map $(\theta, \varphi) \mapsto (q, p)$ is $f(\theta, \varphi) \sin\theta > 0$ for $\theta \in]0, \pi[$ which holds by definition of the spherical coordinate chart on S^2. Thus, the coordinates (q,p) provide a coordinate chart for S^2.

Solution Alternative: We can also find the required coordinates by following the steps of Moser's proof of Darboux' theorem:

Consider a point in the sphere with coordinates $(\theta_0, \varphi_0) \in]0, \pi[\times]0, 2\pi[$ and denote $f_0 = f(\theta_0, \varphi_0)$. We define the forms $\Omega_1 := \Omega(\theta_0, \varphi_0) = f_0 \sin\theta_0 \, d\theta \wedge d\varphi$ and

$$\Omega' := \Omega_1 - \Omega = (f_0 \sin\theta_0 - f(\theta, \varphi)\sin\theta)\, d\theta \wedge d\varphi.$$

By restricting to a neighborhood U_1 of (θ_0, φ_0) we can write $\Omega' = \mathbf{d}\alpha$ where

$$\alpha = \left(\int_{\theta_0}^{\theta} (f_0 \sin\theta_0 - f(\beta, \varphi)\sin\beta) d\beta \right) d\varphi.$$

The lower limit of integration is chosen so that $\alpha(\theta_0, \varphi_0) = 0$. For $t \in [0, 1]$ we define

$$\Omega_t := \Omega + t\Omega' = ((1-t)f(\theta, \varphi)\sin\theta + t f_0 \sin\theta_0)\; d\theta \wedge d\varphi.$$

This form is non-degenerate (the coefficient in front of $d\theta \wedge d\varphi$ is strictly positive) in a neighborhood $U_2 \subset U_1$ of (θ_0, φ_0). In this neighborhood we define the time dependent vector field X_t by the equation $\mathbf{i}_{X_t}\Omega_t = -\alpha$. A calculation shows that

$$X_t(\theta, \varphi) = \left(\frac{\int_{\theta_0}^{\theta} (f(\beta, \varphi)\sin\beta - f_0 \sin\theta_0) d\beta}{(1-t)f(\theta, \varphi)\sin\theta + t f_0 \sin\theta_0} \right) \frac{\partial}{\partial\theta}.$$

Notice that $X_t(\theta_0, \varphi_0) = 0$; so there exists a neighborhood $U \subset U_2$ around (θ_0, φ_0) where the integral curves of X_t are defined for time at least 1. Let $(\bar{\theta}, \bar{\varphi}) \in U$ and let $(\theta(t), \varphi(t))$ be the integral curve of X_t satisfying $(\theta(0), \varphi(0)) = (\bar{\theta}, \bar{\varphi})$. According to Moser's proof, the desired coordinate transformation in U (that makes Ω into a constant) is given by $(\bar{\theta}, \bar{\varphi}) \mapsto (\theta(1), \varphi(1)) := (q, p)$.

A direct calculation shows that the above mentioned integral curve $(\theta(t), \varphi(t))$ satisfies:

$$\varphi(t) = \bar{\varphi}, \qquad \int_{\theta_0}^{\theta(t)} ((1-t)f(\beta, \bar{\varphi})\sin\beta + t f_0 \sin\theta_0)\, d\beta = \int_{\theta_0}^{\bar{\theta}} f(\beta, \bar{\varphi})\sin\beta \, d\beta.$$

Putting $q = \theta(1), p = \varphi(1)$ we find:

$$p = \bar{\varphi}, \qquad q = \theta_0 + \frac{1}{f_0 \sin\theta_0} \int_{\theta_0}^{\bar{\theta}} f(\beta, \bar{\varphi})\sin\beta \, d\beta.$$

In these coordinates we have:

$$\Omega = f_0 \sin\theta_0 \, dq \wedge dp.$$